Ergodic properties and equilibrium of one-dimensional self-gravitating systems

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Recent studies of one-dimensional self-gravitating systems have raised new questions about their ergodic properties, what defines equilibrium for these systems, and their ability to reach a state of thermal equilibrium in a finite time. Earlier studies of small-N systems ($N<11$) using Lyapunov exponents have shown that stable regions exist in the phase space which prevent these systems from thermalizing. Here we investigate several small-N systems with specific initial states in an attempt to answer some of the questions of ergodicity and relaxation toward equilibrium which have been sparked by recent large-N ($N=64$) simulations. Using time averages of the specific particle energy deviations from equipartition, we see similar peaks occurring in the data for small-N simulations as have been reported for large N. Instead of being an indication of the onset of equilibrium, these peaks may indicate regions of the phase space where the system resides for extremely long periods of time. The existence of sticky regions in the phase space in both small- and large-N systems raises questions about the structure of the phase space, relaxation, and the appropriateness of various tests of equilibrium. Here we show that equipartition is not sufficient to remove fundamental doubts concerning the system's ergodic properties. [S1063-651X(97)01908-9]

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I. INTRODUCTION

The determination of the time scale for the relaxation of an isolated, gravitationally bound system, such as a galaxy or globular cluster continues to be a central problem in stellar dynamics. The one-dimensional self-gravitating system (OGS) has been used for several decades as a simple model to study relaxation in gravitating systems. Since the gravitational force in the one-dimensional (1D) system is uniform, the equations of motion are simple algebraic equations which can be easily and rapidly solved on a computer. This allows for very long-time computer simulations with little loss of numerical accuracy. In addition, simulations of one-dimensional systems do not suffer from some of the difficulties encountered in three dimensions (e.g., singularities, evaporation), but as a consequence may not have as strong a connection with the real universe. Computer simulations of the 1D systems show that they tend to progress through various quasiequilibrium states as they evolve from arbitrary initial conditions. These quasiequilibrium states often last for very long times, and are approximately stationary. The decay of fluctuations within the quasiequilibrium state is referred to as microscopic relaxation to distinguish it from the longer, macroscopic, time scale for achieving thermal equilibrium. Recent dynamical simulations have demonstrated that the relaxation time to equilibrium (if it exists) from arbitrary initial conditions is orders of magnitude greater than had been predicted.

The one-dimensional self-gravitating system was originally suggested as a model for the motion of stars perpendicular to the plane of a highly flattened galaxy [1,2]. Since then, others have used the system to study Lynden-Bell's theory of violent relaxation [3], and the usefulness of the Vlasov theory for systems with a large number of components [4]. Researchers have intensely investigated the ergodic properties of the OGS and its possible relaxation toward equilibrium [5–13].

Whether the OGS is capable of reaching a true equilibrium state, and, if it can, when and how it occurs, have been persistent questions in astrophysics since the 1960s. Much work has been done in studying the dynamics of small-to-medium-sized systems ($N<30$) looking for evidence of strong ergodic behavior which would result in eventual thermal equilibrium [5–13]. The study of Lyapunov exponents and the decay of correlations in time have been used in attempts to determine when and if thermalization occurs. A positive Lyapunov exponent would guarantee the existence of a mechanism for the OGS to come to equilibrium in a finite time starting from arbitrary initial conditions if the energy hypersurface has a single ergodic component. Despite extensive research, the ergodic properties of the OGS are still not well understood, and many fundamental questions remain.

There are however, several facts that are known about the system. For example, many of the basic equilibrium properties of the OGS were derived by Rybicki [14]. In his paper, Rybicki derived the single-particle equilibrium distribution function using both the canonical and microcanonical ensembles. In the large-N limit, keeping the total energy and mass fixed, these discrete particle distribution functions reduce to the Vlasov forms previously derived by Camm using methods taken from plasma physics [2]. Vlasov dynamics may be valid, and have been applied to very large collections of gravitating masses such as galaxies and globular clusters in which the particles can be treated in a mean-field approximation.

In addition, very long-lived core-halo structures in μ space [$\mu=(x,v)$] are known to occur in the OGS following an initial period of violent relaxation from arbitrary initial conditions. In the real universe these types of structures are

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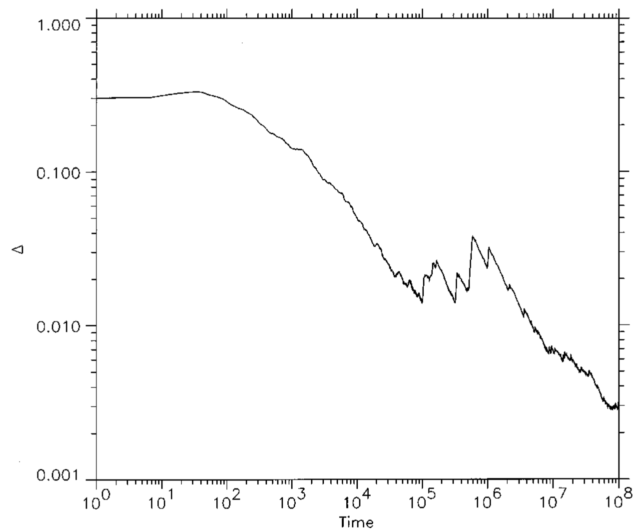

FIG. 1. $\Delta(t)$ for a 64-particle system with the waterbag initial conditions showing an ''equilibrium'' peak as defined by Tsuchiya, Konishi and Gouda followed by a general trend toward zero. Time is measured in units of τ.

observed in configuration space (e.g., in globular clusters), and are typified by a dense, massive core in near-equilibrium conditions surrounded by a halo of high-energy particles (stars) that interact only weakly with the core material.

In a recent series of papers, Tsuchiya, Konishi, and Gouda conducted long-time simulations of medium- to large-N systems ($N=16-512$) to observe evidence of thermalization [15,16]. They constructed a global measure $\Delta(t)$ of the deviation from equipartition. As a working hypothesis, they assumed for the OGS that, if $\Delta(t)$ approaches zero, the system is ergodic and comes to equilibrium. Using time averages of the specific particle energy deviations from the infinite time average equipartition energy value $[\Delta(t)]$, they tracked the evolution of the system starting from arbitrary initial conditions. In general, $\Delta(t)$ will approach zero on an infinite time scale, a necessary condition for a system that achieves equilibrium. However, in systems that cannot freely roam the energy surface (nonergodic), $\Delta(t)$ may still tend to zero if certain symmetry conditions are satisfied, even if the system is confined to a subset of the energy surface, or it may approach a constant different from zero if the symmetry is broken.

All of the simulations reported by Tsuchiya, Konishi, and Gouda showed a characteristic large peak in $\Delta(t)$ some time after the system had microscopically relaxed. This jump was attributed to the onset of a transition from an arbitrary quasiequilibrium state to true macroscopic equilibrium. The behavior of $\Delta(t)$ was assumed to continue its trend toward zero after this initial peak. Both the microscopic and macroscopic relaxation time scales were empirically found to be proportional to N with different proportionality constants. The population dependence of these time scales was recently shown to follow directly from diffusion models developed and studied by the authors [17–19].

Figure 1 shows the results of a dynamical simulation we carried out on a 64-particle system with the ''waterbag'' initial conditions (constant density in μ space) of Refs. [15,16]. Several other discrete jumps in $\Delta(t)$ can be seen to occur after the initial ''equilibrium'' peak. Within the time period simulated, $\Delta(t)$ seems to continue its trend toward zero, although with small but noticeable peaks continuing to occur. The discovery of these secondary peaks provided the initial motivation for this study.

In this paper we first show that $\Delta(t)\rightarrow 0$ does not imply ergodicity, and then verify this by giving an example where $\Delta(t)\rightarrow 0$ for a known nonergodic system. In this way we show that $\Delta(t)\rightarrow 0$ is a necessary, but not sufficient, condition for ergodicity. Next, we examine the behavior of $\Delta(t)$ in several small population versions of the OGS ($N=4$, 6, 8, and 10) and qualitatively compare the results to a large ($N=64$) system whose initial state was sampled directly from the canonical ensemble. For small systems ($N<11$), stable structures have been shown to exist in the phase space which prevent the system from fully exploring the complete energy hypersurface [12,13]. Very long-time simulations of these small systems yield more complete information about the structure of the phase space which sheds light on the phase space structure of larger systems. For several systems, $\Delta(t)$ seems to approach a nonzero constant. At first (small N) the time to flatness increases rapidly with N, but then seems to saturate at about 10^8 time units.

II. DESCRIPTION OF THE SYSTEM

The discrete one-dimensional gravitating system is a collection of N planar sheets of constant mass density σ infinite in, say, the x and y directions, that can move along the z direction under their mutual gravitational attraction. Only gravitational forces are considered, thus the sheets do not collide but merely pass through (cross) one another. Since all sheets have the same mass density, during a crossing the sheets simply exchange accelerations and therefore experience a discrete jump, while the sheet velocities remain continuous functions of time. Between crossings the sheets simply undergo uniform acceleration produced by the inhomogeneity of the mass distribution. Because the system is isolated, momentum conservation allows us to fix the center of mass and set the total momentum to zero. The acceleration experienced by the jth sheet from the left depends only on the difference between the number of sheets (mass) to the right and the left, and is given by

$$A_j=2\pi G\sigma(N-2j+1), \tag{1}$$

where G is the universal gravitational constant. The energy of a system of sheets is constant and is given by

$$E=\tfrac{1}{2}\sigma\sum_{j=1}^{N} v_j^2+2\pi G\sigma^2\sum_{j<i}|x_i-x_j|, \tag{2}$$

where v_j and x_j are the velocity and position of the jth sheet, respectively. If the particles are ordered and labeled consecutively from left to right (i.e., $x_{j+1}>x_j$), the energy can then be expressed as

$$E=\tfrac{1}{2}\sigma\sum_{j=1}^{N} v_j^2+2\pi G\sigma^2\sum_{j=1}^{N-1} j(N-j)(x_{j+1}-x_j). \tag{3}$$

To see this, consider the work done by gravity in reducing the distance $x_{j+1}-x_j$ to zero, while keeping the distances between all other sheets constant. This work is equivalent to bringing two sheets together of mass $j\sigma$ and $(N-j)\sigma$ (i.e., the mass on the left and right). Repeat this process for each pair of sheets until all the sheets are coincident. The potential energy is given by the sum of those terms, and the total energy is that shown in Eq. (3). It is customary to define the characteristic period of a sheet in the system as $t_c = (G\rho_0/\pi)^{-1/2}$, where ρ_0 is the equilibrium mass density evaluated at the origin. This represents a typical period of oscillation of a particle in the system. Because the potential energy is a homogeneous function of the coordinates of first degree, all dependence on parameters can be removed by introducing convenient units as follows [14]:

$$L=\frac{2E}{3\pi GM^2},\quad V=\left[\frac{4E}{3M}\right]^{1/2},\quad T=\left[\frac{1}{\pi MG}\right]\left[\frac{E}{3M}\right]^{1/2}, \tag{4}$$

where L is the length, V is the velocity, T is the time, G is the universal gravitational constant, and E and M are the total system energy and mass, respectively. Dimensionless units of acceleration, velocity, position, and time are then given by

$$A\rightarrow a=\frac{A}{2\pi MG},\quad V\rightarrow v=\frac{V}{2}\left(\frac{3M}{E}\right)^{1/2}$$

$$X\rightarrow x=\left(\frac{3\pi GM^2}{2E}\right)X,\quad t\rightarrow\tau=\frac{t}{T}. \tag{5}$$

We will adopt these units in the remainder of the paper. In these units the characteristic period $t_c \approx 2\pi\tau$, and $2\pi G=1$. With the total mass fixed, $M=1$, the average energy of the system is then $E=0.75$.

Obviously, the system can also be viewed as a collection of particles (mass points) moving in one dimension, each of mass $m=1/N$, where m replaces σ in Eqs. (1)–(3), and we will use this language freely. In dimensionless units the acceleration of the jth particle with the ordered labeling is

$$a_j=\frac{1}{N}(N-2j+1). \tag{6}$$

Rybicki derived an exact expression for the canonical and microcanonical single particle distribution function for the discrete system [14]. The canonical single-particle distribution function takes the form

$$f_c(p,x)=\theta_c(p)\rho_c(x), \tag{7}$$

with

$$\theta_c(p)=\left[\frac{\beta N}{2\pi\sigma(N-1)}\right]^{1/2}\exp\left[\frac{-\beta Np^2}{2\sigma(N-1)}\right], \tag{8}$$

$$\rho_c(x)=N\beta\lambda\sum_{q=1}^{N-1}A_q^N\exp[-N\beta\lambda q|x|], \tag{9}$$

where

$$\beta=\frac{1}{kT},\quad \lambda=2\pi G\sigma^2,\quad A_q^N=\frac{q(-1)^{q-1}[(N-1)!]^2}{(N-1-q)!(N-1+q)!}. \tag{10}$$

k is Boltzmann's constant, T is the temperature, and N is the total number of particles in the system.

In the Vlasov limit the total energy and mass of the system is held constant while the number of particles in the system is allowed to approach infinity. The probability density in $\mu=(x,v)$ space of the resulting continuous fluid satisfies the Vlasov equation

$$\frac{\partial f}{\partial t}=-v\frac{\partial f}{\partial x}-a\frac{\partial f}{\partial v}, \tag{11}$$

In this limit, the velocity and position equilibrium distribution functions reduce to

$$\theta(v)=\pi^{-1/2}\exp(-v^2)\quad\text{(velocity)},$$

$$\rho(x)=\tfrac{1}{2}\operatorname{sech}^2(x)\quad\text{(position)}, \tag{12}$$

where x and v are in dimensionless units.

III. STATISTICAL MEASURES

The state of a dynamical system is given by a single point in the $2N$-dimensional phase space (Γ space), whose trajectory is governed by Hamiltonian dynamics as the state of the system changes. Since initial conditions are never exactly known, a probability density can be defined that indicates a range of possible states that the system can occupy. For a conservative system such as the OGS, the total energy is a constant and defines an isolating integral. Therefore, the phase-space trajectory of the system will be restricted to the energy hypersurface S_E defined by the Hamiltonian.

For a system to approach equilibrium from an initial arbitrary state it must exhibit two important properties in the phase space: ergodicity and mixing. A system is ergodic if the phase-space average of a dynamical quantity is equal to the time average. Ergodic flow can exist on the energy surface only if there are no other isolating integrals that will restrict the trajectories. The above properties imply that all areas of the energy surface are equally accessible, all states on the energy surface are equally probable, and the system will spend equal time in equal areas on the energy surface. Ergodicity is a necessary, but not a sufficient, condition for a system to approach equilibrium from an arbitrary initial state. To approach equilibrium, the system must also exhibit the property of mixing. The decay of correlations in time is a necessary consequence of mixing behavior in phase space [20]. Even so, mixing theorems generally tell us nothing about the rate of approach to equilibrium or the mechanisms to achieve it. Determining if specific systems are ergodic and mixing is typically very difficult and has been proven exactly for only a few systems. In addition the role of dimension (number of degrees of freedom) in shaping the ergodic properties and macroscopic behavior of dynamical systems is not well understood. Dimension may play a critical role in the phase space evolution of gravitational systems. Systems with long range gravitational forces, where every particle continuously ''feels'' every other particle, may evolve in phase space differently than systems with short range near-neighbor forces such as gases.

As one measure of phase space ergodicity, Tsuchiya *et al.* have used a test for equipartition of energy in the OGS [15,16]. The energy per unit mass (specific energy) of the jth particle in the system is given by

$$\varepsilon_j = \frac{1}{2} v_j^2(t) + \frac{1}{N} \sum_{i=1}^{N} |x_i(t) - x_j(t)|. \tag{13}$$

Following Boltzmann, if a system is completely ergodic over the energy surface, the value of a macroscopic observable quantity is simply the time average of the corresponding microscopic operator evaluated over an infinite time

$$B = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\; b(x(t)) = \overline{b}. \tag{14}$$

Equipartition of energy, the equal division of the system energy among its constituent particles in a dynamical system over an indefinitely long time, is based only on the equality of simple time averages. Thus, for a system of pairwise interacting particles, if the time average kinetic energy is the same for each particle, and the time-averaged interaction potential is the same for every pair, equipartition is guaranteed. If, further, the interaction potential is a homogeneous function of order ν, the virial theorem fixes the relationship between the time averaged kinetic and potential energy [21]. Ergodicity is not required here either, only the existence of bounds for the coordinates and momenta.

For a system that achieves equipartition and has order $\nu = 1$, the infinite time average of the specific energy (energy per unit mass), ε_i will assume a unique value for all i,

$$\varepsilon_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \varepsilon_i(t) dt = \varepsilon_0 \equiv \frac{5E}{3}. \tag{15}$$

During a simulation, we can measure the degree of deviation from equipartition by computing the quantity

$$\Delta(t) \equiv \frac{1}{\varepsilon_0} \left(\frac{1}{N} \sum_{i=1}^{N} [\overline{\varepsilon_i}(t) - \varepsilon_0]^2 \right)^{1/2}, \tag{16}$$

where $\overline{\varepsilon_i}(t)$ is the averaged value of the specific energy up to a time t.

For systems that exhibit ergodic behavior on the energy surface, the quantity $\Delta(t)$ should tend to zero in the infinite time limit. However, as we have seen, equipartition of energy is not sufficient to prove ergodicity, and therefore is not a conclusive measure of the approach to equilibrium. According to Tsuchiya, Konishi, and Gouda, $\Delta(t)$ tends to zero approximately as $t^{-1/2}$. If there exist other isolating integrals that restrict the trajectories of the system on the energy surface (segmented phase space) or regions exist where the trajectories tend to remain for very long times (sticky areas), then $\Delta(t)$ should converge to a nonzero constant (flatten) in a finite time. This ''time to flatness'' of $\Delta(t)$ should increase with increasing N, and would be evidence of nonergodicity and long-time correlations that prevent proper energy sharing among the particles. This asymmetry is known to occur in the three-particle version of the OGS, and is associated with interparticle correlations and the clumping of particles in the phase space [22].

To illustrate these ideas, we briefly consider an entirely different system consisting of mass points on the line which interact solely via elastic collisions. Figures 2 and 3 show dynamical simulations of this one-dimensional, colliding, mass point gas using Δ as a measure of equipartition [23]. Figure 2 shows Δ vs the cumulative number of collisions (collision number) for a system of ten equal mass particles. It is well known that a finite system of equal mass points is not ergodic since, on collision, they just exchange velocity. However, we can see that the system is capable of efficiently sharing energy between particles so that Δ is a decreasing function (on average) tending toward zero, indicating equipartition of energy. Figure 3 shows Δ vs the collision number for a system of ten particles with masses randomly drawn from a uniform distribution.

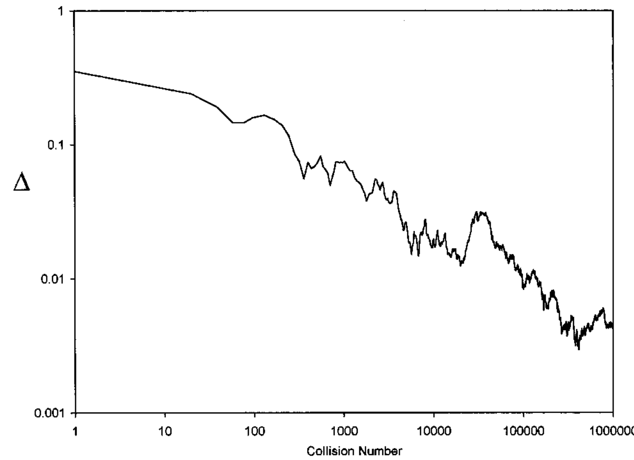

FIG. 2. Δ plotted vs the cumulative number of collisions (collision number) for a ten particle one-dimensional colliding mass point gas with all particles having equal mass. For this known nonergodic system, Δ decreases on average as a function time.

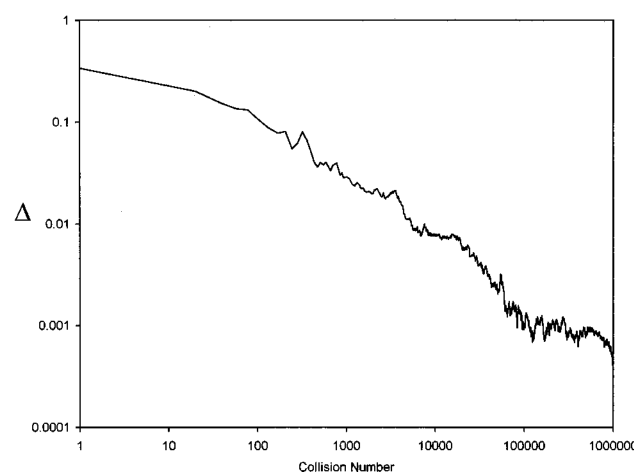

FIG. 3. Δ plotted vs the cumulative number of collisions (collision number) for a ten particle one-dimensional colliding mass point gas with masses drawn from a random distribution. This system has been shown to approach the equilibrium distribution in the long time limit (see Ref. [23]). Here also Δ decreases on average as a function time as expected.

Numerical simulations have shown that this system relaxes to the theoretical equilibrium distribution function on a finite time scale. Comparisons between the numerical and theoretical distributions were made using a cumulative χ^2 statistic with a level of significance much less than 0.01 [23]. This implies that this system is both ergodic and mixing. We see in Fig. 3 that $\Delta(t)$ tends toward zero as expected. Therefore, as we discussed earlier, we see in these dynamical simulations that equipartition of energy by itself does not guarantee ergodicity.

IV. SMALL-*N* DYNAMICAL SIMULATIONS AND RESULTS

Previous studies of the OGS indicate very interesting phase-space properties for small-N systems. In particular, studies by Reidl and Miller (RM) showed the possible existence of a critical population ($N \geqslant 11$) in which 1D systems become chaotic [12,13]. An estimate for the thermalization time for an $N=11$ system was $3.1\times 10^7 t_c (\approx 1.95\times 10^8 \tau)$ as a lower bound. Their work also suggested a rapid increase in the thermalization time with an increase in system population.

Lyapunov exponents were calculated as a function of time for each mode using two nearby trajectories. Convergence to a single Lyapunov exponent would indicate that the phase space consists of a single ergodic component (i.e., unsegmented). However, the simulations were not run for long enough to confirm positively the convergence of the Lyapunov exponents to a single value for the two different modes. Convergence at a much longer time was inferred by extrapolation of the data. Even if the OGS is not ergodic and mixing at $N=11$, these experiments seem to indicate a fundamental change in behavior at the critical population. A flattening of $\Delta(t)$ to a nonzero constant in a finite time for $N=11$ would indicate nonergodic behavior and segmentation of the phase space. However, if no flattening were observed in a finite simulation time, the results would be inconclusive, since we have seen that equipartition of energy is not a sufficient measure of an approach to equilibrium.

To test these ideas we first investigated several small-N systems ($N=4$, 6, 8, 10, and 11) using the statistical methods described in Sec. III. The initial conditions are small perturbations from known periodic orbits (breathing mode and mode 1) derived by RM and described in Refs. [12] and [13]. The breathing mode is highly unstable, and small perturbations from periodicity grow very rapidly for all system populations due to the high probability of multiple particle encounters. Mode 1 orbits are much more stable than the breathing mode since they contain only two-particle encounters, but appeared to become unstable for even the smallest perturbation at a population of $N=11$.

In Figs. 4–8 we plot Δ vs time in units of τ for each of the above systems. The approximate time to flatness, where visible, is indicated in the caption. Although $N<11$ systems are not ergodic, prominent peaks in Δ are apparent in each figure. In addition, the $N=4$, 6, and 8 systems seem to develop flat regions after a time which increases with N. No flatness has developed in the $N=10$ and 11 systems after a time $t=4\times 10^8 \tau$. This increase in time to flatness, if indeed Δ in these higher-dimensional systems exhibits flatness, is not completely unexpected as the system population increases toward $N=11$.

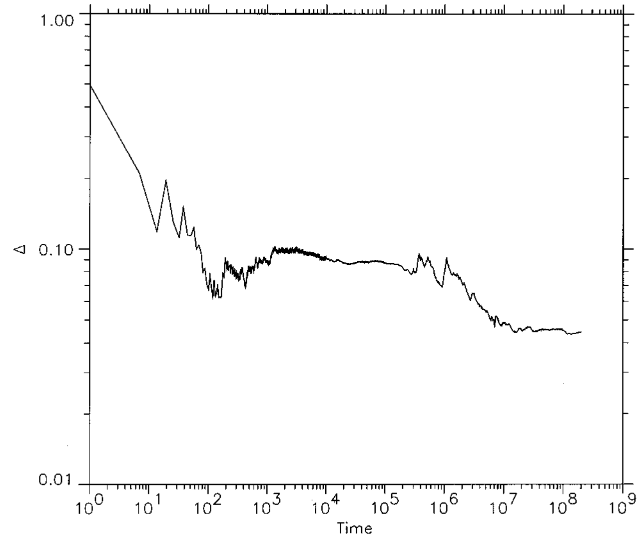

FIG. 4. $\Delta(t)$ for a four-particle system whose initial condition is a small perturbation from the breathing mode. Several peaks are seen, and a flattening occurs around $t=1\times 10^7$. Time is measured in units of τ.

From Figs. 7 and 8, it is too difficult to tell if $\Delta(t)$ will continue its general downward trend to zero or will flatten out. What is clear is that $\Delta(t)$ decreases extremely slowly, even very far out in time where any effects of fluctuations from equipartition should be extremely small.

To determine if $\Delta(t)$ plots can distinguish between stable and unstable regions in the phase space, we chose the initial condition as a small perturbation from the highly stable mode 1 periodic orbit. Figure 9 shows a typical plot for an $N=8$ system. In common with the earlier plots, which arise in unstable regions of the phase space, here also we see one

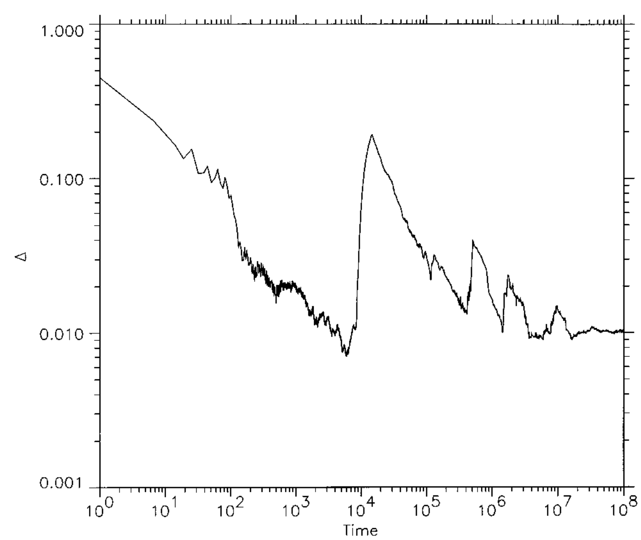

FIG. 5. $\Delta(t)$ for a six-particle system whose initial condition is a small perturbation from the breathing mode. An initial large peak is seen followed by several smaller peaks. The general decline toward zero in $\Delta(t)$ is interrupted by a flattening that occurs around $t=2\times 10^7$. Time is measured in units of τ.

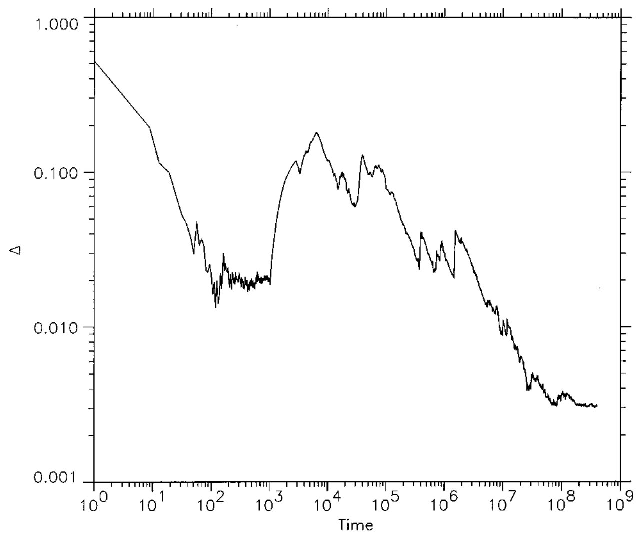

FIG. 6. $\Delta(t)$ for an eight-particle system whose initial condition is a small perturbation from the breathing mode. Several large peaks are seen followed by several smaller peaks. The general decline toward zero in $\Delta(t)$ is interrupted by a flattening that occurs around $t=1.5\times10^8$. Time is measured in units of τ.

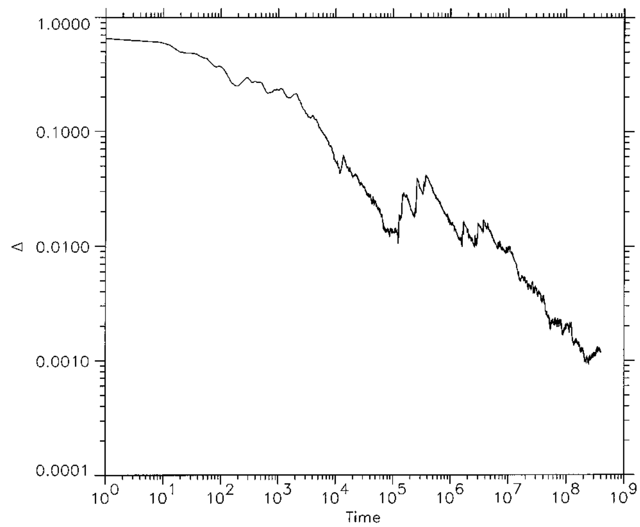

FIG. 8. $\Delta(t)$ for an 11-particle system whose initial condition is a small perturbation from the breathing mode. Two initial peaks are seen followed by several smaller peaks. $\Delta(t)$ continues its general trend toward zero with no obvious flattening. A large jump upward is seen around $t=2.5\times10^8$. Time is measured in units of τ.

large and several smaller peaks followed by a continued reduction in $\Delta(t)$.

V. EQUILIBRIUM INITIAL CONDITIONS AND RESULTS

If the hypothesis of Tsuchiya, Konishi, and Gouda that the occurrence of the large peak in $\Delta(t)$ signifies the onset of equilibrium is correct, then it should not arise if the initial state is drawn from the equilibrium ensemble. To construct an equilibrium state, it is tempting to simply sample the equilibrium μ-space distribution, Eq. (12), directly. For $N=64$ it is easily shown that this is close to the exact single-particle density, Eqs. (7)–(10) derived by Rybicki [14]. However, this procedure would ignore the correlations between particles. Therefore the resulting point in phase space would not be characteristic of true equilibrium. To avoid this difficulty we carefully initialized a 64-particle system by directly sampling the canonical ensemble. The method we selected respects all of the interparticle correlations and therefore is a highly probable representation of equilibrium for the OGS.

The canonical distribution is given by $[(1/Z_N)\exp(-\beta H)]$, where Z_N is the partition function and H is the system Hamiltonian. To sample the configuration space, we note from Eq. (3) that the potential energy of the system can be written as a sum over nearest-neighbor distances. Thus, in the canonical ensemble, these nearest-

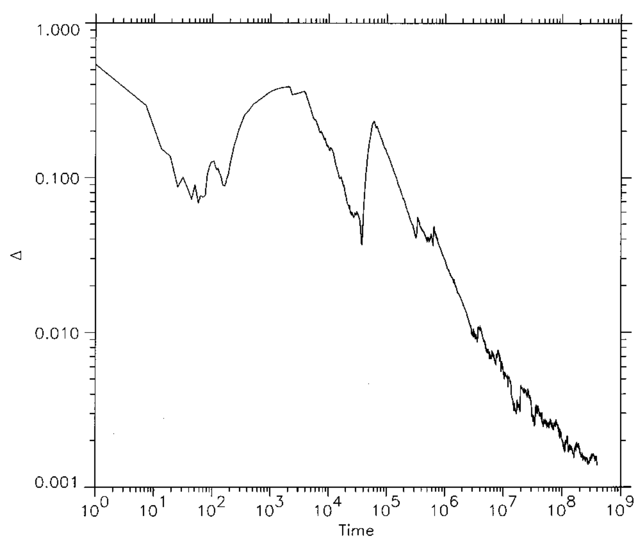

FIG. 7. $\Delta(t)$ for a ten-particle system whose initial condition is a small perturbation from the breathing mode. Two large peaks are seen followed by several smaller peaks. $\Delta(t)$ continues its general trend toward zero with no obvious flattening, although a gradual reduction of slope becomes apparent around $t=2\times10^7$. Time is measured in units of τ.

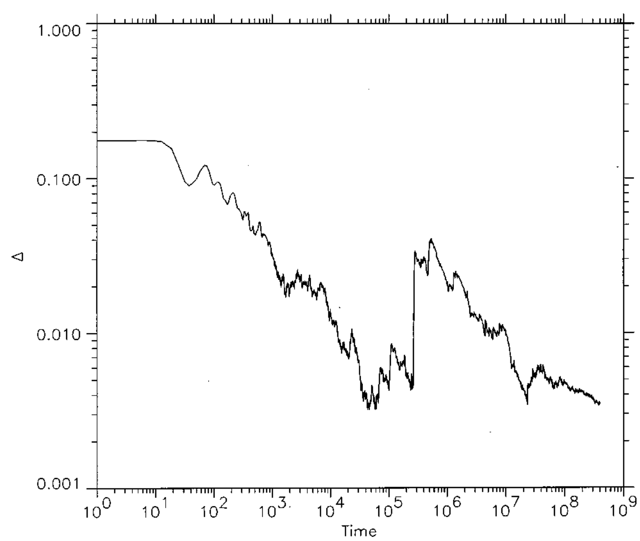

FIG. 9. $\Delta(t)$ for an eight-particle system whose initial condition is a small perturbation from mode I. One large peak is seen after a considerable time has past. $\Delta(t)$ continues its general trend toward zero with no obvious flattening, although a significant reduction in slope is seen around $t=4\times10^7$. Time is measured in units of τ.

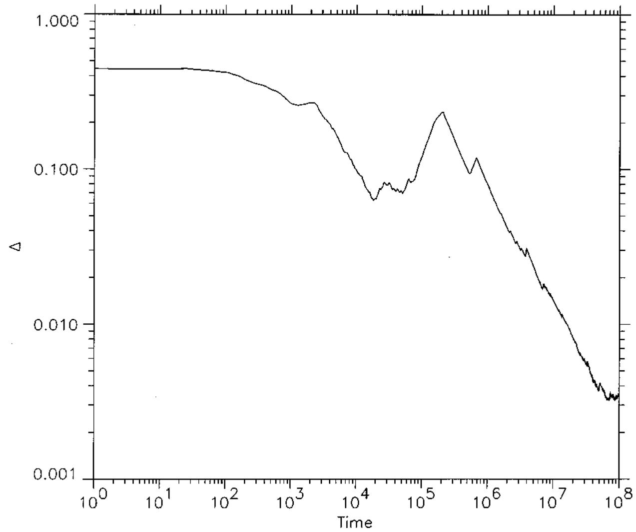

FIG. 10. $\Delta(t)$ for a 64-particle system with the initial condition sampled from the canonical ensemble. An ''equilibrium'' peak is seen around $t=1\times10^5$ followed by a general trend toward zero. Time is measured in units of τ.

neighbor distances are distributed exponentially. After sampling the exponentials, the particle positions are then shifted to fix the center of mass at the origin. The particle velocities are set by first freely sampling a single particle's velocity from a Gaussian distribution, and then using Levy's method [24] to obtain the velocities of the other $N-1$ particles of the ordered system from the canonical ensemble under the additional constraint that the total momentum vanishes. Further details can be found elsewhere [14,18].

The motivation for sampling the canonical ensemble is to create an initial condition which is a typical equilibrium state. This is not to imply that the system evolves isothermally, i.e., while interacting with a heat bath. Once the energy is set in a simulation, it remains fixed. However, since the potential energy of the system is a homogeneous function of the coordinates of degree 1, the ergodic properties of a phase space orbit do not depend on the energy [14], so that the canonical ensemble is also appropriate for finite N.

Figure 10 shows $\Delta(t)$ up to a time $t=1\times10^8\tau$ for a simulation with the initial equilibrium system described above. We see here that the $\Delta(t)$ resulting from the equilibrium system has all of the central features, with the exception of flattening, of both small and large N.

VI. DISCUSSION AND CONCLUSIONS

The determination of the ergodic properties of an arbitrary dynamical system with many degrees of freedom is extremely difficult with current tools. There are a few systems where definite statements are possible. Commonly known examples of integrable systems are colliding point masses on the line (equal masses), coupled linear oscillators, and the Toda lattice, whereas some known ergodic systems are an assortment of billiards (stadium, Sinai, wedge, etc.). Proofs of ergodicity, either analytic or numerical, for systems with arbitrary degrees of freedom are extremely rare. Because the dynamics of the OGS is only slightly more complex than point particles on the line, it might seem that the ergodic properties should be well known, but this is not the case.

The papers by Tsuchiya, Konishi, and Gouda suggest that computations of $\Delta(t)$ and its asymptotic behavior provide a measure of the attainment of equilibrium, which requires strong ergodic properties. Thus, if their hypothesis that the first large peak occurring in plots of Δ vs time signifies the onset of equilibrium were correct, this would provide such a tool. Unfortunately, our studies of both small and large systems do not support this.

We have shown that a majority of the dynamical simulations of both stable and unstable nonergodic small-N versions of the OGS, as well as a 64-particle system prepared in an equilibrium state, exhibit a large central peak in $\Delta(t)$ and other peaks which decrease in size on average as time progresses. Rather than indicating the onset of equilibrium, these peaks in $\Delta(t)$ probably represent areas of the phase space in which the system resides for very long periods of time (sticky regions), since these occur in systems with known stable structures on the energy hypersurface in phase space. The nature of the time average for $\Delta(t)$ would tend to reduce the magnitude of peaks that occur later in time even if these peaks were caused by similar events.

Equipartition of energy, i.e., the vanishing of $\Delta(t)$ in the asymptotic ($t\to\infty$) limit, has classically been associated with the approach to equilibrium. Since the classic studies of nonlinear oscillators carried out by Fermi, Pasta, and Ulam [25], energy equipartition has been identified with equilibrium. In this paper we have shown that blind faith in equipartition as a litmus test for equilibrium is not justified. Lack of equipartition, or ''flattening'' [i.e., the failure of $\Delta(t)$ to converge to zero], certainly demonstrates nonergodic behavior. Flattening indicates a segmented energy surface and broken symmetry on actual trajectories. These features were demonstrated in the various dynamical simulations described here. Simulations of multiple colliding mass points on a line show $\Delta(t)$ approaching zero for both ergodic (nonequal mass) and nonergodic (equal mass) versions of the multiple colliding mass point system. For the OGS systems, we found either flattening or equipartition depending on the initial conditions. For the $N=8$ system, equipartition of energy seems to occur even in a stable region of the phase space. In common with the earlier simulations of Tsuchiya, Konishi, and Gouda, all systems were run for a period of time between $10^8\tau$ and $10^9\tau$. It is of course possible that flattening could occur in any of these systems on longer time scales and hints of this may be seen in the 64-particle simulations.

In summary, a number of studies have shown a decided lack of an approach to equilibrium. There is evidence that the OGS resides for some time in states which look like equilibrium and then later drifts away [9]. In addition, studies of time correlations did not show conclusive convergence to zero on very long-time scales [10]. Systems initialized in a stationary, nonequilibrium, Vlasov state (waterbag) have remained in this state for long times, and have provided no evidence of thermalization or evolution to another stationary state [26]. At the same time, dynamical studies of small systems have shown the presence of a chaotic segment of the energy hypersurface whose measure increases with N. That is, the system looks more like a thermodynamic system as N increases. Examples of this behavior are the increasing

Kolmogorov entropy found by Benettin, Froeschle, and Scheidecker [27], the increase of the diffusion rate of particles out of clusters [28], and the approach of the velocity and position distributions to the predictions of the microcanonical ensemble [29,30].

In earlier work we showed that the mode 1 orbit changes from stable to unstable at $N=11$. A possible scenario for this system is that stable periodic orbits always exist in the phase space for any finite N, but that their number decreases with N. This will have to be explored later. In this current work we have shown that recent measures proposed to test relaxation to equilibrium are insufficient, but do indicate the existence of sticky regions in the phase space where the system is localized for long time periods. At this time, we believe that there is no definitive proof of ergodicity in the OGS.

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